Discrete Mathematics 310 (2010) 1253–1258

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Structure of recurrent sequences of median filters

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ARTICLE INFO

Article history:
Received 4 August 2008
Received in revised form 2 December 2009
Accepted 4 December 2009
Available online 23 December 2009

Keywords:
Median filter
Root
Recurrent sequence

ABSTRACT

In this paper, we first obtain a fundamental result on the structure of recurrent sequences of median filters. Based on this result, we then classify recurrent sequences and discuss their periodicity.

1. Introduction

In this paper, k is a fixed integer and $\mathbf{Z}$ is the set of integers. Let x be a real sequence, indexed by $\mathbf{Z}$. For each integer n, we denote by $x^{(1)}(n)$ the median value of the following $2k+1$ numbers:

$$x(n-k), x(n-k+1), \dots, x(n), \dots, x(n+k-1), x(n+k).$$

By this operation, the sequence x is transformed into a new one $x^{(1)}$, which is called the *median filter* of x with window width $2k+1$. If we apply the median filter to $x^{(1)}$ (with window width $2k+1$) again, another sequence $x^{(2)}$ is obtained. We denote by $x^{(p)}$ the sequence obtained from x by p applications of the median filter.

If $x^{(1)} = x$, then x is a *root*; if $x^{(1)} \neq x$ and there exists $s \geq 2$ such that $x^{(s)} = x$, then x is a *recurrent sequence*. Suppose $\Delta \subset \mathbf{Z}$ and x is a real sequence. If for each $n \in \Delta$, $\lim_{p\to\infty} x^{(p)}(n)$ exists and is finite, then x is *locally convergent* with respect to the median filter on Δ; if $\lim_{p\to\infty} x^{(p)}(n)$ is a real number $r(n)$ for each $n \in \mathbf{Z}$, then x is *convergent* with respect to the median filter, denoted by $x^{(p)} \to r$ (as $p \to \infty$).

In [4] we proved that a sequence is convergent with respect to the median filter with window width $2k+1$ if and only if the sequence is locally convergent on a segment of length $2k-1$ in the sequence. This shows that not all sequences are convergent with respect to the median filter. Let x be a real sequence not convergent with respect to the median filter. In [3] we proved that both $\{x^{(2p)}\}_{p\geq 1}$ and $\{x^{(2p-1)}\}_{p\geq 1}$ are convergent, and if $x^{(2p)} \to \alpha$ and $x^{(2p-1)} \to \beta$, then both α and β are recurrent sequences of the median filter. A natural question is: What structure do the limiting sequences α and β have? In fact, any recurrent sequence of the median filter is a limiting sequence. Recurrent sequences were introduced by Tyan [2], who conjectured that all recurrent sequences are binary (that is, only two distinct numbers occur in the sequence). Zhou [5] and Brandt [1] independently showed that the conjecture is true and proved that $x^{(2)} = x$ if x is a recurrent sequence. In this paper, we will first give a fundamental result on recurrent sequences. Then, using this result, we will classify recurrent sequences and discuss their periodicity.

For simplicity, we set some notions as follows. Let x be a real sequence. Consider $n_1, n_2 \in \mathbf{Z}$ with $n_1 < n_2$. Let $[n_1, n_2]$ denote $\{i : n_1 \leq i \leq n_2\}$. We call $x[n_1, n_2]$ a *segment with length* $n_2 - n_1 + 1$ in x.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2009.12.005

Let x be a recurrent sequence of median filter with window width $2k+1$ whose elements are a or b with $a < b$. Let

$$y(n) = \begin{cases} 1, & \text{if } x(n) = b, \\ -1, & \text{if } x(n) = a, \end{cases} \quad n \in \mathbf{Z}.$$

By the definition of median filter with window width $2k+1$, it is obvious that

$$x^{(1)}(n) = a \quad \text{if and only if } y^{(1)}(n) = -1; \qquad x^{(1)}(n) = b \quad \text{if and only if } y^{(1)}(n) = 1.$$

Therefore, for simplicity, we restrict our attention to recurrent sequences whose elements are -1 or 1.

2. A fundamental result on recurrent sequences

First we give a known result about recurrent sequences.

Proposition 1 ([1,5]). *If x is a recurrent sequence of the median filter with width $2k+1$, then x is binary and $x^{(2)} = x$. Moreover, any segment with length $k+1$ in x is not a constant segment.*

The main result in this paper is the following theorem.

Theorem 1. *Suppose that both x and y are recurrent sequences of the median filter with window width $2k+1$. If there exists an integer n_0 such that $x(j) = y(j)$ and $x^{(1)}(j) = y^{(1)}(j)$ for $n_0 \le j \le n_0 + 2k - 2$, then $x = y$. Moreover, the conclusion is not true for segments of length less than $2k-1$.*

We first prove the following Lemma.

Lemma 1. *Suppose that α is a recurrent sequence of the median filter with window width $2k+1$. If, for some integer n, $\alpha(n) = 1$ and $\alpha(n+1) = -1$, then*

$$\alpha^{(1)}(n-k) = 1, \qquad \alpha^{(1)}(n+1+k) = -1, \quad \textit{and} \quad \sum_{j=n+1-k}^{n+k} \alpha^{(1)}(j) = 0.$$

Proof. Since $\alpha(n) = 1$ and $\alpha(n+1) = -1$, by Proposition 1 we have $\alpha^{(2)}(n) = 1$ and $\alpha^{(2)}(n+1) = -1$. Thus, by the definition of the median filter with window width $2k+1$,

$$\sum_{j=n-k}^{n+k} \alpha^{(1)}(j) \ge 1, \qquad \sum_{j=n+1-k}^{n+1+k} \alpha^{(1)}(j) \le -1. \tag{1}$$

Therefore

$$\alpha^{(1)}(n-k) - \alpha^{(1)}(n+1+k) = \sum_{j=n-k}^{n+k} \alpha^{(1)}(j) - \sum_{j=n+1-k}^{n+1+k} \alpha^{(1)}(j) \ge 2. \tag{2}$$

Since $\alpha^{(1)}(j) \in \{1, -1\}$ for any integer j, by (2) we have $\alpha^{(1)}(n-k) = 1$ and $\alpha^{(1)}(n+1+k) = -1$. Again from (1) it follows that $\sum_{j=n+1-k}^{n+k} \alpha^{(1)}(j) = 0$. $\square$

Remark. If α is recurrent, then also $\alpha^{(1)}$ is recurrent. From Proposition 1, $\alpha^{(2)} = \alpha$. Thus, applying the lemma to $\alpha^{(1)}$ gives that if, for some integer n, $\alpha^{(1)}(n) = 1$ and $\alpha^{(1)}(n+1) = -1$, then

$$\alpha(n-k) = 1, \qquad \alpha(n+1+k) = -1, \quad \text{and} \quad \sum_{j=n+1-k}^{n+k} \alpha(j) = 0.$$

Proof of Theorem 1. Without loss of generality, assume $n_0 = 1-k$. In this case we have

$$x[1-k, k-1] = y[1-k, k-1], \qquad x^{(1)}[1-k, k-1] = y^{(1)}[1-k, k-1].$$

To prove the theorem, we first prove that

($\mathcal{A}$) $x(k) = y(k)$, $x^{(1)}(k) = y^{(1)}(k)$;
($\mathcal{B}$) $x(-k) = y(-k)$, $x^{(1)}(-k) = y^{(1)}(-k)$.

By symmetry (negating indices), it suffices to prove ($\mathcal{A}$).

Since $x^{(1)}$ and $y^{(1)}$ are recurrent sequences of the median filter with window width $2k+1$, it suffices to prove only $x^{(1)}(k) = y^{(1)}(k)$.

Note that also $-x$ and $-y$ are recurrent sequences of the median filter with window width $2k+1$. Hence we may assume that $x(-1) = 1$. Let m be the largest index n such that all entries of $x[-1, n]$ equal 1. Since x is a recurrent sequence, from Proposition 1 we have

$$-1 \le m \le k-2.$$

Case 1. $m = -1$.

In this case, $x(0) = y(0) = -1$ and $x(-1) = y(-1) = 1$. By Lemma 1 we have $x^{(1)}(k) = -1$ and $y^{(1)}(k) = -1$. Thus $x^{(1)}(k) = y^{(1)}(k)$.

Case 2. $m = 0$.

In this case, $x(0) = y(0) = 1$ and $x(1) = y(1) = -1$. By Lemma 1 we have $\sum_{j=1-k}^{k} x^{(1)}(j) = 0$ and $\sum_{j=1-k}^{k} y^{(1)}(j) = 0$. Since $x^{(1)}[1-k, k-1] = y^{(1)}[1-k, k-1]$, we have $x^{(1)}(k) = y^{(1)}(k)$.

Case 3. $1 \le m \le k-2$.

In this case, we have that $x(m) = y(m) = 1$ and $x(m+1) = y(m+1) = -1$. By Lemma 1,

$$\sum_{j=m-k+1}^{m+k} x^{(1)}(j) = 0, \qquad \sum_{j=m-k+1}^{m+k} y^{(1)}(j) = 0.$$

Since $1-k < m-k+1 < k-1$ and $m+k > k$, the equality $x^{(1)}[1-k, k-1] = y^{(1)}[1-k, k-1]$ implies that

$$\sum_{j=k}^{m+k} x^{(1)}(j) = \sum_{j=k}^{m+k} y^{(1)}(j),$$

so

$$\sum_{j=k}^{m+k} [x^{(1)}(j) - y^{(1)}(j)] = 0.$$

If we put $\mu(j) = \frac{1}{2}(x^{(1)}(j) - y^{(1)}(j))$, then the above formula may be written as

$$\sum_{j=k}^{m+k} \mu(j) = 0, \tag{3}$$

where $\mu(j) \in \{-1, 0, 1\}$.

Assume on the contrary that $x^{(1)}(k) \ne y^{(1)}(k)$. In this case, $\mu(k) \ne 0$. Let $E = \{k, k+1, \ldots, m+k\}$. In addition, let

$$E_0 = \{i \in E : \mu(i) = 0\}, \qquad E_1 = \{i \in E : \mu(i) = \mu(k)\}, \qquad E_2 = \{i \in E : \mu(i) = -\mu(k)\}.$$

By (3), we have

$$0 = \sum_{i \in E} \mu(i) = \sum_{i \in E_0} \mu(i) + \sum_{i \in E_1} \mu(i) + \sum_{i \in E_2} \mu(i) = (|E_1| - |E_2|)\mu(k),$$

Since $\mu(k) \ne 0$, we have $|E_1| = |E_2|$. Since $k \in E_1$, $|E_1| \ge 1$. Therefore $|E_2| \ge 1$; that is, E_2 is not empty. Let i_0 be an element of E_2. Thus $\mu(i_0) = -\mu(k)$.

Now we consider the following two cases.

Case 1. $\mu(k) = 1$. In this case, $y^{(1)}(k) = -1$ and $\mu(i_0) = -1$. From $\mu(i_0) = -1$ it follows that $y^{(1)}(i_0) = 1$. Therefore, by the definition of the median filter with window width $2k+1$,

$$\sum_{j=0}^{2k} y(j) < 0, \qquad \sum_{j=i_0-k}^{i_0+k} y(j) > 0.$$

Since $1 \le m \le k-2$ and $k < i_0 \le m+k$, we have $0 < i_0 - k < 2k < i_0 + k$. Hence

$$\sum_{j=i_0-k}^{i_0+k} y(j) - \sum_{j=0}^{2k} y(j) = \sum_{j=2k+1}^{i_0+k} y(j) - \sum_{j=0}^{i_0-k-1} y(j) > 0. \tag{4}$$

Since $k < i_0 \le m+k$, we have $0 \le i_0 - k - 1 \le m-1 < k-1$. By the definition of m, we have $x(j) = 1$ for $0 \le j \le i_0 - k - 1$. Again $x[1-k, k-1] = y[1-k, k-1]$, so we obtain $y(j) = 1$ for $0 \le j \le i_0 - k - 1$. Therefore

$$\sum_{j=0}^{i_0-k-1} y(j) = i_0 - k.$$

By (4),

$$\sum_{j=2k+1}^{i_0+k} y(j) > i_0 - k.$$

Now there are only $i_0 - k$ items in the segment $y[2k+1, i_0+k]$, and $y(j) \in \{1, -1\}$. Thus we get a contradiction. Therefore, we have $x^{(1)}(k) = y^{(1)}(k)$.

Case 2. $\mu(k) = -1$. In this case, $x(k) = -1$ and $\mu(i_0) = 1$. By an argument as that of Case 1, we can also prove that $x^{(1)}(k) = y^{(1)}(k)$. This completes Case 2 and hence also the proof of $\mathcal{A}$.

Now, using $\mathcal{A}$ and $\mathcal{B}$, we have

$$x[-k, k] = y[-k, k] \quad \text{and} \quad x^{(1)}[-k, k] = y^{(1)}[-k, k].$$

Again by (A) and (B), we also have

$$x[-k-1, k+1] = y[-k-1, k+1] \quad \text{and} \quad x^{(1)}[-k-1, k+1] = y^{(1)}[-k-1, k+1].$$

We conclude that $x = y$.

Now we show that the conclusion does not hold for segments of length less than $2k-1$. To do so, it is sufficient to examine the following sequences.

Let x be a periodic sequence with period $4k-2$, defined by

$$x[1, 4k-2] = \{\underbrace{1, 1, \ldots, 1}_{k-1} \underbrace{-1, -1, \ldots -1}_{k-1}, 1, \underbrace{1, 1, \ldots, 1}_{k-1} \underbrace{-1, -1, \ldots, -1}_{k}\}.$$

It is easy to verify that $x^{(1)}$ is also periodic with period $4k-2$, defined by

$$x^{(1)}[1, 4k-2] = \{\underbrace{-1, -1, \ldots, -1}_{k-1}, \underbrace{1, 1, \ldots, 1}_{k-1}, 1, \underbrace{-1, -1, \ldots, -1}_{k-1}, \underbrace{1, 1, \ldots, 1}_{k-1}, -1\},$$

and that $x^{(2)} = x$.

Let y be a periodic sequence with period $4k-2$, defined by

$$y[1, 4k-2] = \{\underbrace{1, 1, \ldots, 1}_{k-1} \underbrace{-1, -1, \ldots, -1}_{k-1}, -1, \underbrace{1, 1, \ldots, 1}_{k-1} \underbrace{-1, -1, \ldots, -1}_{k-1}, 1\}.$$

It is easy to verify that $y^{(1)}$ is also periodic with period $2k-2$, defined by

$$y^{(1)}[1, 4k-2] = \{\underbrace{-1, -1, \ldots, -1}_{k-1}, \underbrace{1, 1, \ldots, 1}_{k-1}, -1, \underbrace{-1, -1, \ldots, -1}_{k-1}, \underbrace{1, 1, \ldots, 1}_{k}\},$$

and that $y^{(2)} = y$. It is clear that $x[1, 2k-2] = y[1, 2k-2]$ and $x^{(1)}[1, 2k-2] = y^{(1)}[1, 2k-2]$, but $x(2k-1) \neq y(2k-1)$. Thus $x \neq y$, even though these sequences and their medians agree on a segment of length $2k-2$.

This completes the proof of Theorem 1. □

3. Classification and periodicity of recurrent sequences

We first give the following definition.

Definition 1. Let x be a recurrent sequence. if $x(n) = x^{(1)}(n)$, then n is a *fixed point* of x.

Lemma 2. *Let x be a recurrent sequence of the median filter with window width $2k+1$. If there exists an integer n_0 such that $n_0, n_0+1, \ldots, n_0+2k-2$ are not fixed points of x, then all points are not fixed points of x.*

Proof. Let $y(n) = -x^{(1)}(n)$ for $n \in \mathbf{Z}$. By the definition of the median filter with window width $2k+1$, we have that y is also a recurrent sequence, and $y^{(1)}(n) = -x(n)$ for each $n \in \mathbf{Z}$. Note that $n_0, n_0+1, \ldots, n_0+2k-2$ are not fixed points of x; that is, $x(n) + x^{(1)}(n) = 0$ for $n_0 \le n \le n_0+2k-2$. Hence

$$y(n) = x(n) \quad \text{and} \quad y^{(1)}(n) = x^{(1)}(n) \quad \text{for } n_0 \le n \le n_0+2k-2.$$

By Theorem 1 we have $y(n) = x(n)$ for $n \in \mathbf{Z}$. Since $y(n) = -x(n)$ for each $n \in \mathbf{Z}$, $x(n) + x^{(1)}(n) = 0$ for any $n \in \mathbf{Z}$; that is, all points are not fixed points of x. □

Lemma 3. *If x is a recurrent sequence of the median filter with window width $2k+1$, then, for each $n \in \mathbf{Z}$, in $n, n+1, \ldots, n+2k-2$ there is at least one not being fixed point of x.*

Proof. Assume on the contrary that there exists $n_0 \in \mathbf{Z}$ such that $n_0, n_0+1, \dots, n_0+2k-2$ are fixed points of x; that is,

$$x^{(1)}(n) = x(n) \quad \text{for } n_0 \le n \le n_0+2k-2.$$

In this case, if $y = x^{(1)}$, then y is also a recurrent sequence, and $y^{(1)} = x^{(2)} = x$. So we have

$$y(n) = x^{(1)}(n) = x(n) = y^{(1)}(n) \quad \text{for } n_0 \le n \le n_0+2k-2.$$

From Theorem 1 it follows that $y = x$. Therefore $x^{(1)} = x$. This contradicts the fact that x is a recurrent sequence of the median filter with window width $2k+1$. Thus for each $n \in \mathbf{Z}$, among $n, n+1, \dots, n+2k-2$ there is at least one not being fixed point of x. □

Theorem 2. *If x is a recurrent sequence of the median filter with window width $2k+1$, then*

(i) *all points are not fixed points of x;*

or

(ii) *there are both fixed points and points not being fixed points in each segment with length $2k-1$ in x. Moreover, the conclusion does not hold for shorter segments.*

Proof. By Lemmas 2 and 3, it is sufficient to show that the conclusion does not hold for shorter segments. To do so, consider the following sequence.

Suppose that z is a periodic sequence with period $4k-2$, defined by

$$z[1, 4k-2] = \{-1, \underbrace{1, 1, \dots, 1}_{k-1}, \underbrace{-1, -1, \dots, -1}_{k-1}, 1, \underbrace{1, 1, \dots, 1}_{k-1}, \underbrace{-1, -1, \dots, -1}_{k-1}\}.$$

It is easy to verify that $z^{(1)}$ is also periodic with period $4k-2$, defined by

$$z^{(1)}[1, 4k-2] = \{-1, \underbrace{-1, -1, \dots, -1}_{k-1}, \underbrace{1, 1, \dots, 1}_{k-1}, 1 \underbrace{-1, -1, \dots, -1}_{k-1}, \underbrace{1, 1, \dots, 1}_{k-1}\},$$

and $z^{(2)} = z$. Since $z(1) = z^{(1)}(1)$ and $z(j) + z^{(1)}(j) = 0$ for $2 \le j \le 2k-1$, there exist fixed points of z, and all points in the segment $z[2, 2k-1]$ are unfixed points of z. Therefore, the conclusion in (ii) does not hold for shorter segments. □

By Theorem 2, we may divide recurrent sequences into two classes: a recurrent sequence satisfying (i) is called a recurrent sequence of category I ; the recurrent sequence satisfying (ii) is called a recurrent sequence of category II.

Next we discuss periodicity of recurrent sequences.

Lemma 4. *Suppose that both x and y are recurrent sequences of the median filter with window width $2k+1$. If there exists an integer n_0, such that $x(n) = y(n)$ for $n_0 \le n \le n_0+4k-2$, then $x = y$.*

Proof. Since $x(n) = y(n)$ for $n_0 \le n \le n_0+4k-2$, by the definition of median filter, we have $x^{(1)}(n) = y^{(1)}(n)$ for $n_0+k \le n \le n_0+3k-2$. Thus from Theorem 1 it follows that $x = y$. □

Theorem 3. *All recurrent sequences of the median filter with window width $2k+1$ are periodic sequences with periods less than 2^{4k-1}.*

Proof. Let x be a recurrent sequence of the median filter with window width $2k+1$. Denote by Λ the class of all $(4k-1)$-tuples z such that

$$z(n) \in \{-1, 1\} \quad \text{for } 1 \le n \le 4k-1.$$

The class Λ is a finite set, with altogether 2^{4k-1} elements.

Let

$$X = \{x[n, n+4k-2] : 1 \le n \le 2^{4k-1}\}.$$

It is clear that $X \subset \Lambda$. Since X has no constant segment, $X \ne \Lambda$. Therefore, $|X| < 2^{4k-1}$. Thus there exist integers p, q with $1 \le p < q \le 2^{4k-1}$ such that $x[p, p+4k-2] = x[q, q+4k-2]$. In this case, put $y(n) = x(n+q-p)$ for $n \in \mathbf{Z}$. Now y also is a recurrent sequence, and

$$x(n) = y(n) \quad \text{for } p \le n \le p+4k-2.$$

By Lemma 4 we conclude that $x(n) = y(n)$ for $n \in \mathbf{Z}$. So

$$x(n) = x(n+q-p) \quad \text{for } n \in \mathbf{Z}.$$

Therefore x is a periodic sequence with period $q-p < 2^{4k-1}$. This completes the proof of Theorem 3. □

Suppose that both x and y are recurrent sequences of category I. If there exists an integer n_0 such that $x(n) = y(n)$ for $n_0 \leq n \leq n_0 + 2k - 2$, then by Theorem 1 we obtain $x = y$. By the use of an argument as that of Theorem 3, we can obtain the following result.

Theorem 4. *All recurrent sequences of category I of the median filter with window width* $2k + 1$ *are periodic sequences with periods less than* 2^{2k-1}.

Acknowledgements

We thank deeply the referees and the editor for their helpful suggestions and comments on the paper, which have improved the presentation. The first author is supported by Shanghai Leading Academic Discipline Project (J50101). The research is supported by Key Disciplines of Shanghai Municipality (S30104) and NNSF of China (60672160).

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